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THE SOLUTION OF BOUNDARY VALUE PROBLEMS OF THE THEORY OF SHELLS WITH AN UNKNOWN BOUNDARY*

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The elliptical boundary value problem with an unknown free boundary is solved using analytic continuation into the domain of independent complex variables. The unknown region of solution is estimated using the method of characteristics based on the Schwarz mapping principle and the correspondence between the solutions of the Cauchy problem in the elliptical and hyperbolic region. A numerical solution of the problem in question is also obtained using Galerkin's method and Green's functions.

Problems of the theory of elasticity with an unknown boundary of the domain of solution were studied in /1, 2/. The zone of influence of a hole in a stretched plate was determined in /2/ from the condition that the right-hand side of the characteristic equation on the contour of a unit circle obtained after the conformal transformation should be analytic. The method of characteristics and the conditions for a solution to exist, determined by the Cauchy-Riemann theorem, were used to obtain the domain of solution of the hyperbolic equation for a toroidal shell /3/. An analogous boundary value problem is solved below for an elliptical domain.

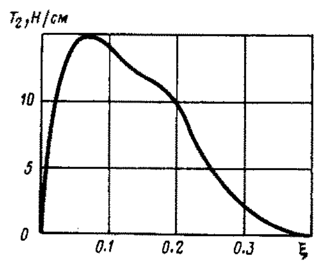

Fig.1

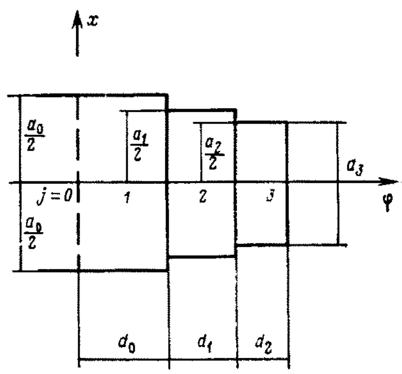

Fig.2

Consider a segment of a torus with a meridional narrow out bounded by the coordinates $0 \leqslant \theta \leqslant \pi/6$ and $0 \leqslant \varphi \leqslant 2\pi$ (θ, φ are the angular coordinates in the meridional and peripheral direction). A force P_z is applied to the edge $\theta_0 = 0$ through a movable disc, the edge $\theta_1 = \pi/6$ is clamped rigidly and remains fixed. The edges $\varphi = 0$ and $\varphi = 2\pi$ are free of external loads. The zone of influence of the edge $\varphi = 0$ determining the domain of solution of the basic partial differential equation is not known, and is found for thin shells using membrane theory. Here the basic equation has the form /4/

$$\frac{1}{R_1 R_2 \sin\theta}\frac{\partial}{\partial\theta}\left[\frac{R_2^2 \sin\theta}{R_1}\frac{\partial U}{\partial\theta}\right] + \frac{1}{R_1 \sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} = 0 \tag{1}$$

where $U = -T_2 R_1 \sin^2\theta$, T_2 are the tensile forces in the peripheral direction and R_1, R_2 are the radii of principal curvatures of the middle surface of the shells. In the case of a toroidal shell Eq.(1) will become

$$\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\varphi^2} = 0, \quad \xi = A^{-1/2}\xi_1 \tag{2}$$

*Prikl.Matem.Mekhan.,51,1,172-175,1987

$$A = \frac{1}{\theta_1 - \theta_0} \int_{\theta_0}^{\theta_1} \frac{\sin^3\theta}{(\lambda + \sin\theta)^3}\, d\theta, \quad \xi_1 = \int_{\theta_0}^{\theta_1} \frac{\sin\theta}{(\lambda + \sin\theta)^3}\, d\theta, \quad \lambda = \frac{d}{R_1},$$

(d is the distance between the axis of symmetry of the toroidal shell and the centre of its meridional cross-section).

Taking into account the fact that the stress state is axisymmetric and continuous outside the zone of influence of the edge $\varphi = 0$, we adopt for (2) the following boundary conditions:

$$U|_{\varphi=0} = 0, \quad U|_L = f(\xi), \quad \partial U/\partial\varphi|_L = 0 \tag{3}$$

Here L is the unknown boundary, $f(\xi)$ is a function of T_2 known from the solution of the similar problem for a shell closed over φ /5/; for a segment $D \leqslant \theta \leqslant \pi/6$ of a torus with $R_1 = 20$ cm, $h = 0.02$ cm, $\lambda = 3.3$ and $P_z = 98$ N, the function $T_2(\xi)$ is shown in Fig.1.

Let us transform Eq.(2) by introducing the following substitution:

$$Y(\xi, \varphi) = f(\xi) - U(\xi, \varphi) \tag{4}$$

$$\frac{\partial^2 Y}{\partial \xi^2} + \frac{\partial^2 Y}{\partial \varphi^2} = F(\xi), \quad F(\xi) = \frac{d^2 f}{d\xi^2} \tag{5}$$

Let us approximate the domain of solution of Eq.(5) with a set of rectangular elements (RE) of dimensions a_j, d_j $(j = 0, 1, 2, \ldots, n)$ (Fig.2).

We shall write the boundary conditions (3) taking into account (4), in the form

$$Y|_{\varphi=0} = f(\xi), \quad Y|_L = 0 \tag{6}$$

On the sides of RE parallel to the φ axis we have

$$dY/d\varphi = 0 \tag{7}$$

In mathematical language problem (5)-(7) is equivalent to the problem of the propagation of electromagnetic waves in a closed system of rectangular waveguides. We specify the source of perturbation at the boundary in the transverse cross-section of the waveguide shown by the dashed line in Fig.2, and consider the propagation and reflection of the waves in the longitudinal direction /6/.

The solutions of (5) obtained by Galerkin's method for each RE have the form

$$Y(x, \varphi) = \sum_{n=0}^{\infty} F_m{}^j (\gamma_j \operatorname{sh} \gamma_j d_j)^{-1} \sin \frac{n\pi}{a_j}\left(x + \frac{a_j}{2}\right), \quad \gamma_j = \frac{n\pi}{a_j} \tag{8}$$

$$F_m{}^j = \frac{\varepsilon_n}{2a_j} \iint_{G(x,\varphi)} S_j(x) \sin \frac{n\pi}{a_j}\left(x + \frac{a_j}{2}\right) \mathrm{H}\, dx\, d\varphi \tag{9}$$

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0, \end{cases} \quad \mathrm{H} = \begin{cases} \operatorname{ch} \gamma_j (d - \varphi') \operatorname{ch} \gamma_j \varphi^0, & \varphi^0 < \varphi' \\ \operatorname{ch} \gamma_j \varphi' \operatorname{ch} \gamma_j (d - \varphi^0), & \varphi^0 > \varphi' \end{cases}$$

$$S_j(x) = \frac{d^2 f_{0j}(x)}{dx^2}, \quad f_{0j}(x) = \sum_{\alpha=1}^{m} B_{j\alpha} \sin \frac{\alpha\pi}{a_j}\left(x + \frac{a_j}{2}\right)$$

Here x is a new variable chosen in such a manner that the φ axis is the axis of symmetry of RE, $S_j(x)$ is Green's function, $B_{j\alpha}$ are unknown coefficients, φ' is the coordinate of the plane of coupling of the RE, and φ^0 is the coordinate of the plane of integration of Green's functions characterizing the electromagnetic fields in the separate RE's. The choice of the function $\eta(\varphi^0, \varphi')$ is determined by the theorem of equivalence, on the basis of which the real sources of electromagnetic field are replaced by equivalent surface currents and integral equations of the type (9) constructed /7/.

The conditions that the solutions (8) match at the boundaries of the RE parallel to the x axis (Fig.2), yield a system of equations. After integrating this system we obtain the coefficients $B_{j\alpha}$ from a system of algebraic equations.

In the well-known problem of electrodynamics the first boundary condition of (6) corresponds to specifying the source of perturbation, and the condition of the minimum field of reflected wave determining the zone of influence of the source reduces to finding a minimum value of the integral

$$I_0 = \int_{-a_0/2}^{a_0/2} \frac{d^2}{dx^2}\left[B_{0\alpha} \sin \frac{\alpha\pi}{a_0}\left(x + \frac{a_0}{2}\right)\right] dx$$

Specifying the consecutive sets of values a_j, d_j, beginning from the smallest possible value and increasing them in discrete steps, we determine I_0. The condition for a minimum of the function I_0 yields the form and size of the unknown boundary L. We see from the numerical calculations carried out for the parameters given earlier, that the maximum size of the domain of solution (in the φ direction) is equal to $\varphi_0^{(1)} = 0.41$ (Fig.2).

We obtain the estimate of the maximum size of the domain of solution of (5) by analytic continuation of the function Y to the domain of the complex variable $\xi = \xi_1 + i\xi_2, \varphi = \varphi_1 + i\varphi_2$. Here (5) corresponds to the wave equation /8/ $\partial^2 Y/\partial\varphi_1^2 - \partial^2 Y/\partial\xi_2^2 = F(\xi)$. Its characteristics are determined by the expression $d\xi_2/d\varphi_1 = \pm 1$, which forms a family of parallel lines parallel to the bisectrices of the coordinate angles whose apices satisfy the relation $\xi_2 \pm \varphi_1 = \text{const}$. From the first boundary condition of (6), by equating ξ_2 to the magnitude of the segment of the known part of the boundary (Fig.1,2) and the characteristic passing through until it intersects the φ_1 axis, we obtain an approximate estimate of the maximum size of the unknown domain of solution of $\varphi_0^{(2)} = 0.396$, i.e. the difference between it and the result obtained earlier using the method of integral equations, does not exceed 3.5%. The comparison shows the possibility of using the method of characteristics to solve elliptical boundary value problems with an unknown boundary in the theory of thin shells.

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VIBRATION OF AN ELASTIC ROD WITH DRY FRICTION ON ITS SIDE SURFACE*

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Steady longitudinal oscillations in a semibounded elastic rod are studied, taking into account "dry" friction on its side surface. An approximate solution is constructed using the method of harmonic linearization /1/ which leads to a boundary value problem for a system of two non-linear equations. The latter can be reduced to the Cauchy problem by a change of variables. Results of numerical computations are given.

We consider the longitudinal oscillations of a weightless one-dimensional elastic rod of constant cross-section, taking into account dry friction on its side surface. Steady oscillations are discussed, unlike in /2/ where a problem with initial data was solved for the case when the end face of the rod was loaded according to special laws. We specify a harmonic perturbation of the deformation at one of its ends and assume the other end (removed to infinity) to be at rest, to obtain the system

$$\rho S\partial^2 u/\partial t^2 = ES\partial^2 u/\partial x^2 - q\,\mathrm{sign}\,(\partial u/\partial t) \tag{1}$$
$$x = 0,\ u = u_0 \cos \omega t;\ x \to \infty,\ u \to 0\ (u_0 \equiv \text{const} > 0)$$

where u, S denote the displacement and the area of transverse cross-section, ω is the

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